

Period-index bounds for arithmetic threefolds

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ABSTRACT. The standard period-index conjecture for Brauer groups of p -adic surfaces S predicts that $\text{ind}(\alpha) \mid \text{per}(\alpha)^3$ for every $\alpha \in \text{Br}(\mathbf{Q}_p(S))$. Using Gabber's theory of prime-to- ℓ alterations and the deformation theory of twisted sheaves, we prove that $\text{ind}(\alpha) \mid \text{per}(\alpha)^4$ for α of period prime to $6p$, giving the first uniform period-index bounds over such fields.

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1. INTRODUCTION

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let R be an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ with fraction field K . Suppose k is semi-finite or separably closed. Let L be an extension of K of transcendence degree 2, and let $\alpha \in \text{Br}(L)$ be a Brauer class. If α has period prime to p then*

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^5$$

and if α has period prime to $6p$ then

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^4.$$

Recall from [27] that k is a semi-finite field if it is perfect and if for every prime ℓ , the maximal prime-to- ℓ extension of k is pseudo-algebraically closed with Galois group \mathbf{Z}_ℓ . Finite fields and pseudo-finite fields are semi-finite. As a special case, we obtain the following result.

Corollary 1.2. *Let S be a geometrically integral surface over a p -adic field K . If $\alpha \in \text{Br}(K(S))$ has period relatively prime to $6p$, then*

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^4.$$

The period of a central simple algebra is its order in the Brauer group and its index is the degree of a Brauer equivalent division algebra. The period divides the index and both numbers have the same prime factors. Results bounding the index in terms of the period have motivated many of the developments in the theory of the Brauer group since the beginning of the 20th century. See [3, Section 4] for a survey of results of this type.

For local and global fields, the index equals the period by Albert, Brauer, Hasse, and Noether (see [13, Remark 6.5.6]). For a finitely generated field of transcendence degree 2 over an algebraically closed field, the index equals the period by de Jong [10] (see also [25]). More generally, Artin conjectured that the index equals the period for every C_2 field, and he proved this for algebras of index a power of 2 or 3, see [2]. For a field of transcendence degree 1 over a local field, the index divides the square of the period by Saltman [32] for algebras of index prime to the characteristic and Parimala and Suresh [30] in general. Analogous results for fields of transcendence degree 1 over higher local fields are established in [26] and subsequently in [16] by other methods. For fields of transcendence degree 2 over a finite field, the index divides the square of the period by [27].

Such results support the following conjecture, cf. [7, Section 2.4].

Conjecture 1.3 (Period-index conjecture). *Let K be a field of transcendence degree n over a field k . For every $\alpha \in \text{Br}(K)$ we have*

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^{n-1+e}$$

where we take $e = 0, 1, 2$ when k is algebraically closed, C_1 , or p -adic, respectively.

Based on this conjecture, we do not expect the period-index bound we achieve in Theorem 1.1 to be optimal. However, this is the first proof of a general period-index bound that is uniform in the period for fields of transcendence degree 2 over a local field. For classes of period a power of 2, bounds on the u -invariant are known to imply uniform bounds for the index in terms of the period; our bounds are still better than what can be attained using known u -invariant results for function fields over p -adic fields [23]. There are also non-uniform period-index bounds for C_i fields due to Matzri [28].

Our approach follows a strategy inspired by Saltman [32]: split the ramification of the Brauer class by a field extension of controlled degree and then use geometry to study the unramified Brauer class on a regular proper model. For the former, we draw on, and expand upon, a development due to Pirutka [31] (see Section 2). After splitting the ramification and using Gabber's refined theory of ℓ' -alterations to reduce to a regular (quasi-semistable) model, we reduce the proof of Theorem 1.1 to the following general result. Given an integral scheme X , we write $\kappa(X)$ for its function field; given $\alpha \in H^2(\kappa(X), \mu_n)$, we write $\text{per}(\alpha)$ and $\text{ind}(\alpha)$ for the period and index of the associated class in $\text{Br}(\kappa(X))$.

Theorem 1.4. *Let R be an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ and with fraction field K . Suppose that X is a connected regular 3-dimensional scheme, flat and proper over $\text{Spec } R$. Let $\alpha \in H^2(X, \mu_n)$ where n is prime to p . Assume that the Brauer class of α is trivial on all proper closed subschemes of the reduced special fibre $X_{0,\text{red}}$ of dimension at most 1. If $\text{ind}(\alpha|_{\kappa(X_i)}) = \text{per}(\alpha|_{\kappa(X_i)})$ for all irreducible components X_i of $X_{0,\text{red}}$, then $\text{ind}(\alpha_{\kappa(X)}) = \text{per}(\alpha_{\kappa(X)})$.*

Note that the hypothesis, that the Brauer class of α is trivial on all proper subschemes of $X_{0,\text{red}}$ of dimension at most 1, is automatically satisfied if k is semi-finite or separably closed (see Lemma 3.1).

The proof of Theorem 1.4 uses the deformation theory of twisted sheaves to reduce the computation of the index of a Brauer class on a regular model to the existence of certain twisted sheaves on the reduced special fibre, which we can assume is a simple normal crossings surface. In the case when the special fibre is smooth, this approach was carried out in [25, Proposition 4.3.3.1]. In the general case, we end up proving a version of de Jong and Lieblich's period-index theorems for simple normal crossings surfaces over separably closed and semi-finite fields, respectively.

It is known that Saltman's theorem is the best possible for p -adic curves. Indeed, examples were given by Jacob and Tignol in an appendix to [32] to this effect. Conjecture 1.3 predicts that for a surface over $\mathbf{C}((t))$ one has $\text{ind}(\alpha) \mid \text{per}(\alpha)^2$, while for a surface over a p -adic field one has $\text{ind}(\alpha) \mid \text{per}(\alpha)^3$. The non-optimality of our results is undoubtedly due to our overly generous splitting of ramification. The approach taken in [27] improves these kinds of bounds at the expense of a layer of stacky complexity.

Outline. In Section 2, we generalize work of Pirutka [31] on splitting the ramification of Brauer classes. Sections 3, 4, and 5 discuss the existence and deformation theory of twisted sheaves on proper models of the function field we consider. Section 6 considers Gabber's refined theory of ℓ' -alterations in the context of splitting ramification. Theorems 1.1 and 1.4 are proved in Section 7.

Notation. If X is a scheme and R is a commutative ring, we denote by $H^i(X, \mu_n)$, $H^i(R, \mathbf{G}_m)$, and so on the corresponding étale cohomology groups.

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2. SPLITTING RAMIFICATION

The results of this section are, for the most part, a generalization and reworking of the results of Pirutka [31] (which themselves live in a tradition of ramification-splitting results due to Saltman [32]). We follow Pirutka's strategy with minor modifications so that it works in mixed characteristic, and we give a somewhat different argument on the existence of rational functions whose roots split ramification.

Our ultimate goal in this section is to show that we can split all of the ramification occurring in the Brauer classes of interest to us with relatively small extensions.

2.1. Local description of ramification. Recall that a regular system of parameters in a regular local ring is a minimal generating set of the maximal ideal. A subsequence of a regular system of parameters is called a *partial* regular system of parameters. Not every regular sequence is a partial regular system of parameters.

Definition 2.1.1. Let R be a regular local ring with fraction field F , and suppose that $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$. We say that α is *nicey ramified* if α is ramified only along a partial regular system of parameters x_1, \dots, x_h and we can write

$$\alpha = \alpha_0 + \sum (u_i, x_i) + \sum_{i,j} m_{i,j} (x_i, x_j)$$

for an unramified class α_0 and some $u_i \in R^\times$ and $m_{i,j} \in \mathbf{Z}$.

More generally, if X is a regular integral scheme with function field F , and $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$, then we say that α is *nicey ramified on X* if it is nicey ramified at every local ring on X .

We will need the following result, prove in the two-dimensional case in [32] and in the equicharacteristic case in [31, Section 3, Lemma 2].

Lemma 2.1.2. *Let X be a regular integral scheme with function field F and let $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$ where ℓ is prime to the residue characteristics of X . Suppose that α is ramified only along a simple normal crossings divisor. Then α is nicey ramified on X .*

Proof. Let R be a local ring of X . By hypothesis, α is ramified only along a partial system of parameters x_1, \dots, x_h . We proceed by induction on h . For $h = 1$, let F_1 be the fraction

field of $R/(x_1)$ and $\beta = (b) \in F_1^\times/F_1^{\times\ell} = H^1(F_1, \mu_\ell)$ the residue of α . Then it follows that β is unramified by considering the Gersten complex

$$H^2(F, \mu_\ell^{\otimes 2}) \rightarrow \bigoplus_{p \in \text{Spec}(R)^{(1)}} H^1(k(p), \mu_\ell) \rightarrow \bigoplus_{q \in \text{Spec}(R)^{(2)}} H^0(k(q), \mathbf{Z}/\ell\mathbf{Z}).$$

For the construction of the complex in this generality, see [20, Section 1]. Since $R/(x_1)$ is a regular local ring, it is a UFD, and we may write $b = \bar{u}_1 \prod \pi_i^{e_i}$ for irreducible elements π_i and a unit \bar{u}_1 of $R/(x_1)$. Since the residue of b at the prime (π_i) is the class of e_i modulo ℓ , it follows that each e_i is a multiple of ℓ , and thus $\beta = (\bar{u}_1) \in F_1^\times/F_1^{\times\ell}$.

Lifting \bar{u}_1 to a unit u_1 of R , it follows that the class $\alpha - (u_1, x_1)$ is unramified on R . In particular, we may write $\alpha = \alpha_0 + (u_1, x_1)$ for α_0 unramified as claimed. For $h > 1$, let $\beta \in F_1^\times/F_1^{\times\ell}$ be the residue of α at the prime (x_1) , as before. Considering the Gersten complex, the residue of β must be cancelled by the residues of α along primes in $R/(x_1)$. In particular, it follows that β can only be ramified along the primes $\bar{x}_2, \dots, \bar{x}_h$ in $R/(x_1)$. Since $R/(x_1)$ is a regular local ring, it is a UFD, and we may represent β by an element $\bar{b} = \bar{u} \prod_{i=2}^h \bar{x}_i^{m_{i,1}}$ with \bar{u} a unit in $R/(x_1)$. In particular, we can lift \bar{b} to $b = u_1 \prod_{i=2}^h x_i^{m_{i,1}}$, where $u_1 \in R^\times$. It follows that

$$\alpha - (b, x_1) = \alpha - (u_1, x_1) - \sum_{i=2}^h m_{i,1}(x_i, x_1)$$

is unramified along (x_1) and only ramified along the primes (x_i) for $i = 2, \dots, h$. By induction, we may write

$$\alpha - (b, x_1) = \alpha_0 + \sum_{j=2}^h (u_j, x_j) + \sum_{j,k \neq 1} m_{j,k}(x_j, x_k),$$

yielding $\alpha = \alpha_0 + \sum (u_i, x_i) + \sum m_{i,j}(x_j, x_k)$ as desired. \square

2.2. Putting ramification in nice position. We'll need the following generalization of [31, Lemma 3], which we prove using toroidal geometry. The standard reference for toroidal geometry is [21], which is written over an algebraically closed base field. However, all the constructions work over an arbitrary base scheme as outlined in [12, IV Remark 2.6].

Lemma 2.2.1. *Let X be a quasi-compact regular scheme of dimension d and suppose that $D \subset X$ is a simple normal crossings divisor. Then there exists a sequence of blowups $f : X' \rightarrow X$ such that*

- (1) X' is regular;
- (2) $f^{-1}D$ is an snc divisor;
- (3) the support of $f^{-1}D$ is a union of d regular (but not necessarily connected) divisors.

Proof. The dual complex $\Delta(D)$ of a simple normal crossings divisor D is a Δ -complex whose vertices I and simplices given by irreducible components of intersections $\cap_{j \in J} D_j$ for subsets $J \subseteq I$. These irreducible components are the strata of the toroidal stratification of X associated to the toroidal embedding $U \hookrightarrow X$, where $U = X \setminus D$. The simplex corresponding to a stratum W has dimension $\text{codim } W - 1$ and is a face of the simplex corresponding to another stratum Y whenever $Y \subseteq W$. As D is a simple normal crossings divisor, $\Delta(D)$ has dimension less than d . Note that the usual fan associated to the toroidal structure on X , as in [21], is the cone over $\Delta(D)$.

Let $f : X' \rightarrow X$ be obtained by sequentially blowing up the strata of D , first blowing up the 0-dimensional strata, then the strict transforms of the 1-dimensional strata, then the strict transforms of the 2-dimensional strata, etc. Then $f^{-1}(D)$ has simple normal crossings, and its dual complex is the barycentric subdivision of $\Delta(D)$. The barycentric subdivision of $\Delta(D)$ is a balanced complex of dimension less than d (see, e.g., [22]), hence its 1-skeleton is d -colorable. If we choose such a coloring, then the irreducible components corresponding to vertices of the same color are disjoint and hence their union is regular. Therefore, we can express $f^{-1}(D)$ as the union of d regular (but not necessarily connected) divisors, as required. \square

2.3. Construction of rational functions for splitting ramification.

Notation 2.3.1. Let $\{V_i = \sum a_{ij}W_{ij}\}_{i \in I}$ be a family of cycles on X . Given a subset $J \subset I$, let V_J denote the naive intersection cycle.

Definition 2.3.2. A collection of irreducible subschemes $\{W_i\}_{i \in I}$ of X *intersects properly* if for every subset $J \subset I$ we have $\text{codim } W_J \geq \sum_{i \in J} \text{codim } W_i$ (using the convention that $\text{codim } \emptyset = \infty$). A collection of cycles $\{V_i\}_{i \in I}$ on X intersect properly if any collection $\{W_i\}_{i \in I}$, where W_i is an irreducible component of the support of V_i for all $i \in I$, intersects properly.

Lemma 2.3.3. *Let X be a scheme. Suppose that $\{W_i\}_{i \in I}$ is a collection of cycles of X that intersect properly. Suppose that $W \subset X$ is an irreducible subscheme such that, for every subset $J \subset I$, the scheme W intersects each irreducible component of W_J properly. Then $\{W_i\} \cup \{W\}$ intersects properly.*

Proof. We omit the proof. □

Now, we prove a lemma which is a direct generalization of the lemma in the correction to [32].

Lemma 2.3.4. *Let X be a quasi-compact regular scheme admitting an ample invertible sheaf and let D_1, \dots, D_d be a collection of regular divisors on X such whose union is a simple normal crossings divisor. Let*

$$\tilde{D}_i = \sum_{j=1}^d m_{i,j} D_j, \quad i = 1, \dots, n$$

be a collection of integer linear combinations of the D_j . Then for $i = 1, \dots, n$ there exist rational functions f_i and divisors E_i on X such that

- (1) $\text{div}(f_i) = \tilde{D}_i + E_i$, and
- (2) *the collection $\{E_i\}_{i=1, \dots, n} \cup \{D_j\}_{j=1, \dots, d}$ intersects properly.*

Proof. We construct the f_i inductively.

Base case $i = 1$: Let P be a (scheme-theoretic) disjoint union of closed points, with one closed point in each irreducible component of D_I for each subset $I \subset \{1, \dots, d\}$. Let R be the semilocal ring of the points of P (which exists since we can put P in a single affine open of X , using the ample invertible sheaf [34, Tag 01PR] and the graded prime-avoidance lemma [11, Section 3.2]). By [6, Chapitre II, Section 5.4, Proposition 5], since finitely generated projective modules over R are free, it follows that each D_i is principal on R . In particular, we may write D_i as the zero locus of a function $x_i \in R$. We then have, upon setting $f_1 = \prod x_j^{m_{1,j}}$, that $(f_1) = \tilde{D}_1 + E_1$ and the support of E_1 contains none of the strata D_I . In particular, since E_1 is a divisor, the codimension of $E_1 \cap D_I$ in D_I is at least 1 as desired.

Induction step: Suppose we have previously defined f_1, \dots, f_{r-1} so that $\text{div}(f_i) = \tilde{D}_i + E_i$ with $\{E_i\}_{i=1, \dots, r-1} \cup \{D_j\}_{j=1, \dots, d}$ intersecting properly. For every pair of subsets $I \subset \{1, \dots, d\}$ and $J \subset \{1, \dots, r-1\}$, consider the intersection $D_I \cap E_J$ and let P be a scheme theoretic union consisting of at least one closed point from each irreducible component of each of these nonempty intersections for every I, J as above. Let R be the semilocal ring at P . As above, we may write D_i as the zero locus of some function $x_i \in R$ on $\text{Spec } R \subset X$, and considering the x_i as rational functions on X , we set $f_r = \prod x_j^{m_{r,j}}$. It follows that $(f_r) = \tilde{D}_r + E_r$ where the support of E_r contains no irreducible component of the strata $D_I \cap E_J$, with $J \subset \{1, \dots, r-1\}$. Therefore $\{E_i\}_{i=1, \dots, r} \cup \{D_j\}_{j=1, \dots, d}$ intersects properly as desired. □

Notation 2.3.5. Let $T \in \text{Mat}_{n,d}(\mathbf{Z})$ be an $n \times d$ matrix. For subsets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, d\}$, let $T_{I,J}$ denote the $|I| \times |J|$ -submatrix of T with rows corresponding to the elements of I and columns corresponding to the elements of J .

Definition 2.3.6. Let ℓ be a prime and n, d be positive integers with $n \geq d$. An $n \times d$ matrix $T \in \text{Mat}_{n,d}(\mathbf{Z})$ is ℓ -Pirutka if for all nonempty subsets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, d\}$, with $|I| - |J| = n - d$, the submatrix $T_{I,J}$ has (maximal) rank $|J|$ modulo ℓ .

Lemma 2.3.7. *Let R be a regular local ring with fraction field F and $\alpha \in H^t(F, \mu_\ell^{\otimes t})$ where ℓ is invertible in R . Let $x_1, \dots, x_n \in R$ be a regular system of parameters and suppose that $\alpha = (u_1, \dots, u_{t-h}, x_1, \dots, x_h)$ with u_i units in R . If $L = F(\sqrt[t]{x_1}, \dots, \sqrt[t]{x_h})$ and S is the integral closure of R in L , then*

- (1) *S is a regular local ring with maximal ideal generated by $\sqrt[t]{x_1}, \dots, \sqrt[t]{x_h}, x_{h+1}, \dots, x_n$,*
- (2) *the class α_L is unramified at each codimension one prime of S .*

Proof. We omit the proof that $S = R[z_1, \dots, z_h]/(z_1^\ell - x_1, \dots, z_h^\ell - x_h)$ and is regular with maximal ideal $\mathfrak{m} = (z_1, \dots, z_h, x_{h+1}, \dots, x_n)$.

Now, for a prime $\mathfrak{P} \subset R$ of height one and a prime $\mathfrak{Q} \subset S$ lying over it with ramification index e , we have a commutative diagram

$$\begin{array}{ccc} H^t(F, \mu_\ell^{\otimes t}) & \longrightarrow & H^{t-1}(R/\mathfrak{P}, \mu_\ell^{\otimes t-1}) \\ \downarrow & & \downarrow e \text{ res} \\ H^t(L, \mu_\ell^{\otimes t}) & \longrightarrow & H^{t-1}(S/\mathfrak{Q}, \mu_\ell^{\otimes t-1}) \end{array}$$

which shows that α_L only ramifies at primes lying over the ramification locus of α . In particular, α_L can only ramify over the primes (z_i) for $i = 1, \dots, h$, which each have ramification index ℓ over (x_i) . Since all residues of α are ℓ -torsion, it follows from the diagram above that α_L is unramified. \square

Lemma 2.3.8. *Let X be a quasi-compact regular scheme of dimension d admitting an ample invertible sheaf and D_1, \dots, D_d be a collection of regular divisors of X whose union is *snc*. Let ℓ be a prime invertible on X and let $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$ be a class ramified only along the union of the D_i . Let $T = (m_{ij})$ be an ℓ -Pirutka $n \times d$ matrix. For each $i = 1, \dots, n$, let $\tilde{D}_i = \sum_{j=1}^d m_{ij} D_j$ and let f_i and E_i be as in Lemma 2.3.4. Let $L = F(\sqrt[t]{f_1}, \dots, \sqrt[t]{f_n})$. Then α_L is unramified.*

Proof. By Lemma 2.1.2, for any point $z \in X$, we have that

$$(2.3.1) \quad \alpha = \alpha_0 + \sum (u_i, x_i) + \sum_{i,j} m_{i,j} (x_i, x_j)$$

for $u_i \in \mathcal{O}_{X,z}^\times$ and x_i local equations for D_i in $\mathcal{O}_{X,z}$.

To show that α becomes unramified in L , it suffices to show that for every point $z \in X$ and each term in (2.3.1), there is a subfield of L where that term becomes unramified with respect to every valuation of L that dominates $\mathcal{O}_{X,z}$. For example, a term of the form (u_i, x_i) will become unramified in an extension of the form $F(\sqrt[t]{g_i})$ where g_i is a local equation for D_i at z ; a term of the form (x_i, x_j) will become unramified in $F(\sqrt[t]{g_i}, \sqrt[t]{g_j})$. (See Lemma 2.3.7.)

We thus seek the following: for each point $z \in X$ and each $j = 1, \dots, d$, an element $g_j \in F$ such that

- (1) g_j is a local equation for rD_j at z , where $r \equiv 1 \pmod{\ell}$;
- (2) $F(\sqrt[t]{g_j}) \subset L$.

Choose J maximal with respect to inclusion so that $z \in D_J$. If $J = \emptyset$ (so that α is unramified over $\mathcal{O}_{X,z}$), then $g_j = 1$ works for all j . Otherwise, choose any $j_0 \in J$; we will find $g_{j_0} \in F$ satisfying conditions (1) and (2) above.

We claim that we can find $I \subset \{1, \dots, n\}$ with $|I| - |J| = n - d$ and $z \notin \cup E_i$. To see this, set

$$I' = \{i \in \{1, \dots, n\} \mid z \in E_i\}.$$

Since $z \in E_{I'} \cap D_J$, it follows by the properness of the intersection that $|I'| + |J| \leq d = \dim(X)$. In particular, there are at most $d - |J|$ indices i such that $z \in E_i$. This means we can find a set of $n - (d - |J|)$ indices i such that $z \notin E_i$. Let I be such a set. Since T is an ℓ -Pirutka matrix, the submatrix $T_{I,J}$ has full rank $|J|$ modulo ℓ , and hence we can find $a_i \in \mathbf{Z}$ for $i \in I$

such that $\sum_{i \in I} a_i m_{i,j} \equiv \delta_{j,j_0} \pmod{\ell}$ for each $j \in J$. Translating in terms of \tilde{D}_i and D_i , this says that there exists $r \equiv 1 \pmod{\ell}$ such that

$$\sum_{i \in I} a_i \tilde{D}_i = r D_{j_0} + \sum_{j \notin J} b_j D_j$$

and therefore

$$\operatorname{div} \left(\prod_{i \in I} f_i^{a_i} \right) = r D_{j_0} + \sum_{j \notin J} b_j D_j + \sum_{i \in I} a_i E_i.$$

Let $g_{j_0} = \prod_{i \in I} f_i^{a_i}$. Since $z \notin E_i$ for all $i \in I$ and $z \notin D_j$ for all $j \notin J$, we find that g_{j_0} is a local equation for $r D_{j_0}$ in $\mathcal{O}_{X,z}$. It is clear that $\sqrt[\ell]{g_{j_0}} \in L$. \square

Theorem 2.3.9. *Let X be a quasi-compact separated regular scheme of dimension d admitting an ample invertible sheaf and $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$ be ramified along a simple normal crossings divisor. If there is an ℓ -Pirutka $n \times d$ matrix, then we can find rational functions $f_1, \dots, f_n \in F$ so that α becomes unramified in $L = F(\sqrt[\ell]{f_1}, \dots, \sqrt[\ell]{f_n})$.*

Proof. By Lemma 2.2.1 we can perform a sequence of blowups to X so as to make the ramification divisor of α contained in an snc divisor that we can write as a union $D_1 \cup \dots \cup D_d$ of regular divisors. Now the result is an immediate application of Lemma 2.3.8. \square

2.4. Some Pirutka matrices.

Example 2.4.1. Pirutka's example is as follows (up to re-interpretation). Consider $n = d^2$, and let T be the $d^2 \times d$ matrix given by d (vertical) copies of the $d \times d$ identity matrix. The condition is now: for every subset of columns J , and subset of rows I of order $d^2 - d + |J|$, we have full rank. But notice that since $|J| \geq 1$, we are always removing fewer than d rows. Since each row of the identity matrix occurs d times, each row of the identity matrix must still occur in the I, J -minor, showing that $T_{I,J}$ has full rank.

Example 2.4.2. The 3×3 matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

considered in [31, Remark 5] is ℓ -Pirutka for all primes $\ell > 3$.

Example 2.4.3. The 4×3 matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

found in [31, Remark 4] is ℓ -Pirutka for all primes ℓ .

Remark 2.4.4. A computer search shows that

- (1) there are no 2-Pirutka 2×2 or larger square matrices, and
- (2) there are no 3-Pirutka 3×3 or larger square matrices.

It is easy to make a 2×2 matrix that is ℓ -Pirutka for all primes $\ell > 2$. This allows one to split the ramification of classes of odd prime period using roots of two rational functions, which reproduce results of Saltman [33] except for classes of period 2.

Question 2.4.5. For what ℓ, n, d do there exist ℓ -Pirutka matrices.

It is clear that if $n \gg d$ this answer is affirmative. We also have the following bound.

Proposition 2.4.6. *If the prime $\ell > \binom{2n-1}{n}$ then there exist ℓ -Pirutka $n \times n$ matrices.*

Proof. First note that an $n \times n$ matrix T is ℓ -Pirutka if and only if all maximal minors of the $n \times 2n$ matrix $A = (I_n | T)$ do not vanish modulo ℓ . We will consider building the matrix $A = (e_1, \dots, e_n, t_1, \dots, t_n)$ by inserting the columns t_i one at a time. For inserting the first column, we simply require that all entries in t_1 do not vanish. Once the first column has been chosen, we require that t_2 avoids $\binom{n+2}{n}$ hyperplanes. When we insert the last column, there are $\binom{2n-1}{n}$ hyperplanes to avoid. This can be done when $\ell > \binom{2n-1}{n}$. \square

Of course this bound is far from sharp. The hyperplanes in the above proof are not in general position.

3. EXISTENCE OF TWISTED SHEAVES ON SIMPLE NORMAL CROSSINGS SURFACES

The purpose of this section is to show that period equals index on simple normal crossings surfaces. We note first that the Brauer groups of the curves occurring in our arguments vanish.

Lemma 3.1. *Let C be a curve over a field k of characteristic $p \geq 0$. If k is separably closed (resp. k is semi-finite and C is proper), then $\mathrm{Br}(C)[n] = 0$ for n prime-to- p (resp. $\mathrm{Br}(C) = 0$).*

Proof. If k is separably closed, then this is [15, Corollaire 1.3]. Thus, assume that k is semi-finite and C is proper over k . Consider the Leray spectral sequence

$$E_2^{st} = H^s(k, R^t \pi_* \mathbf{G}_{m,C}) \Rightarrow H^{s+t}(C, \mathbf{G}_{m,C})$$

for the morphism $\pi : C \rightarrow k$. The only possible contributions to $H^2(C, \mathbf{G}_{m,C})$ are

$$\begin{aligned} H^0(k, R^2 \pi_* \mathbf{G}_{m,C}), \\ H^1(k, R^1 \pi_* \mathbf{G}_{m,C}) &\cong H^1(k, \mathbf{Pic}_{C/k}), \\ H^2(k, R^0 \pi_* \mathbf{G}_{m,C}) &\cong H^2(k, \mathbf{G}_m). \end{aligned}$$

The last vanishes because k is semi-finite. To analyze the second term, let $\tilde{C} \rightarrow C$ be the normalization of the largest reduced subscheme C_{red} of C . By [5, Corollary 9.2.11], there is an exact sequence

$$0 \rightarrow G \rightarrow \mathbf{Pic}_{C/k} \rightarrow \mathbf{Pic}_{\tilde{C}/k} \rightarrow 0$$

of étale sheaves over k , where G is a connected linear algebraic group. Moreover, since \tilde{C}/k is smooth (since k is perfect), there is (for example, by [5, Proposition 3]) an exact sequence

$$0 \rightarrow \mathbf{Pic}_{\tilde{C}/k}^0 \rightarrow \mathbf{Pic}_{C/k} \rightarrow A \rightarrow 0,$$

where A is an étale sheaf on k with $A(k^s) \cong \mathbf{Z}^r$, where r is the number of irreducible components of C . In fact, A becomes constant as soon as each component of \tilde{C} acquires a rational point. As it is enough to prove that $\mathrm{Br}(C)[q] = 0$ for each prime q , we may replace k with its maximal prime-to- q extension, which is pseudo-algebraically closed (so A is in particular isomorphic to the constant sheaf \mathbf{Z}). In particular, every G -torsor and $\mathbf{Pic}_{\tilde{C}/k}^0$ -torsor admits a rational point. Moreover, as $H^1(k, \mathbf{Z}^r) = 0$, it follows that $H^1(k, \mathbf{Pic}_{C/k}) = 0$. It remains to prove that $H^0(k, R^2 \pi_* \mathbf{G}_{m,C}) = 0$. But, the stalk of $R^2 \pi_* \mathbf{G}_{m,C}$ is isomorphic to $H^2(C_{k^s}, \mathbf{G}_m)$, where k^s is the separable closure of k . Since k^s is algebraically closed (as k is perfect), this group vanishes by [15, Corollaire 1.2]. \square

Remark 3.2. There is also a proof that uses Tsen's theorem (resp. class field theory) to treat the regular case and then deduces the general case by deformation from points and a Moret-Bailly-type formal gluing argument, but we omit the argument here.

Remark 3.3. The conclusion that $\mathrm{Br}(C)[n] = 0$ for n prime-to- p cannot be improved to $\mathrm{Br}(C) = 0$ without assuming that k is algebraically closed. If k is separably closed but not algebraically closed, then $\mathrm{Br}(k[x])$ is non-zero. This example appears already in Auslander and Goldman [4, Theorem 7.5]. Consider the Artin-Schreier extension L of $k(x)$ defined by $y^p - y - x = 0$. The ring $k[x, y]/(y^p - y - x)$ is easily seen to be smooth over k , and hence it

is the integral closure of $k[x]$ in L . Since k is not separably closed, there is an element $w \in k$ such that $w \notin k^p$. The algebra

$$k[x]\langle y, z \rangle / (y^p - y - x, z^p - w, zy - yz - z)$$

defines an Azumaya algebra over $k[x]$. For more details, see Gille and Szamuely [13, Section 2.5]. This also explains why the full Brauer group is not \mathbf{A}^1 -homotopy invariant.

The following lemma shows that the only obstruction to extending an \mathcal{X} -twisted locally free sheaf on a curve C inside a surface X is whether or not the determinant extends. It is a direct generalization to the twisted setting of [10, Lemma 5.2], although the proof is slightly different owing to the fact if the μ_n -gerbe \mathcal{X} is non-trivial, then we cannot make use of an \mathcal{X} -twisted line bundle on X .

Lemma 3.4. *Let C be a proper curve in a regular quasi-projective 2-dimensional scheme X over a field k of characteristic $p \geq 0$, and fix a μ_n -gerbe $\mathcal{X} \rightarrow X$, where n is prime to p . Suppose that \mathcal{X} has index n and that the Brauer class of \mathcal{X} vanishes on every proper curve in X , e.g., k is separably closed or semi-finite by Lemma 3.1.*

Suppose that V is a locally free \mathcal{X} -twisted sheaf on C of rank n with $\det V = L|_C$, where $L \in \text{Pic}(X)$. Then, possibly after taking a finite prime-to- n extension of k , there exists a locally free \mathcal{X} -twisted sheaf W on X such that $W|_C \cong V$ and $\det(W) \cong L$.

Proof. Since \mathcal{X} has index n , there is a locally free \mathcal{X} -twisted sheaf E of rank n . Choose an ample line bundle $\mathcal{O}_X(1)$ on X . To prove the Lemma, we will use the following Claim.

Claim 3.5. There is an integer m , a proper curve $D \subset X$ with $\dim(D \cap C) = 0$ in the linear system $|L(mn) \otimes \det(E^\vee)|$, and an invertible \mathcal{X} -twisted sheaf M on D such that there is an exact sequence of \mathcal{X}_C -twisted sheaves

$$0 \rightarrow E(-m)|_C \rightarrow V \rightarrow M|_{C \cap D} \rightarrow 0.$$

We prove the lemma first assuming the claim. Let

$$\gamma \in \text{Ext}_C^1(M|_{C \cap D}, E(-m)|_C)$$

be the corresponding (non-zero) extension class. Our goal is to lift γ to $\text{Ext}_X^1(M, E(-m))$. Given any open subset $U \subseteq X$ containing C , there is an exact sequence

$$\text{Ext}_U^1(M|_{U \cap D}, E(-m)|_U) \rightarrow \text{Ext}_C^1(M|_{C \cap D}, E(-m)|_C) \rightarrow \text{Ext}_U^2(M|_U, E(-m)(-C)|_U).$$

Now,

$$\mathcal{E}xt_U^0(M|_{U \cap D}, E(-m)(-C)|_U) = 0$$

since $M|_{U \cap D}$ is a torsion sheaf, while

$$\mathcal{E}xt_U^2(M|_{U \cap D}, E(-m)(-C)|_U) = 0$$

because $M|_{U \cap D}$ is a locally free sheaf on a curve in U and hence has cohomological dimension 1. From the local-to-global ext spectral sequence, it follows that

$$\text{Ext}_U^2(M|_{U \cap D}, E(-m)(-C)|_U) \cong H^1(U, \mathcal{E}xt_U^1(M|_{U \cap D}, E(-m)(-C)|_U)).$$

If we further choose U to be such that $U \cap D$ is affine and $\dim(X \setminus U) = 0$, then this latter group vanishes since $\mathcal{E}xt_U^1(M|_{U \cap D}, E(-m)(-C)|_U)$ is supported on D and so

$$H^1(U, \mathcal{E}xt_U^1(M|_{U \cap D}, E(-m)(-C)|_U)) \cong H^1(U \cap D, \mathcal{E}xt_U^1(M|_{U \cap D}, E(-m)(-C)|_U)|_D) = 0,$$

by Serre's vanishing theorem for the cohomology of a quasi-coherent sheaf on an affine variety. It follows that γ lifts to an extension

$$0 \rightarrow E(-m)|_U \rightarrow \tilde{V} \rightarrow M|_{U \cap D} \rightarrow 0$$

on U such that $\tilde{V}|_C \cong V$. Let W be $j_*\tilde{V}$, where $j : U \rightarrow X$ is the inclusion. Then, W is reflexive since $X - U$ has codimension 2. By construction it restricts to V on C , and since S is regular and 2-dimensional, W is locally free.

The determinant of W is

$$\det(E(-m)) \otimes \det(M) \cong \det(E)(-mn) \otimes \det(M).$$

Since M is a locally free \mathcal{X} -twisted sheaf of rank 1 on D , $\det(M) \cong \mathcal{O}_X(D)$ (see [25, Proposition A.5]). But D was chosen to be in the class of the linear system associated to $L(mn) \otimes \det(E^\vee)$. It follows immediately that $\det(W) \cong L$, as desired.

Now we prove Claim 3.5. For sufficiently large m , a general map in $\mathrm{Hom}_C(E|_C, V(m))$ is injective and has a cokernel which is an invertible \mathcal{X} -twisted sheaf supported on a general member of the linear system $|N|_C$, where $N|_C = \det(V(m)) \otimes \det(E|_C^\vee)$. (We suppress the fact that N depends on m in the notation.) This follows from [24, Corollary 3.2.4.21]; if k is finite, to use the required Bertini theorem we can take arbitrarily large finite prime-to- n extensions of k to ensure the existence of rational points avoiding the “forbidden cone” (as any open subset of affine space over an infinite field contains rational points). By assumption, the line bundle $N|_C$ is the restriction of the line bundle $N = L(mn) \otimes \det(E^\vee)$ on X . For sufficiently large m , N is ample, and a general member of $|N|$ restricts to a general member of $|N|_C$. We let D be a general regular member of $|N|$ such that $D \cap C$ is the support of an injective map $E|_C \rightarrow V(m)$ with invertible \mathcal{X} -twisted cokernel. By hypothesis, the Brauer class of \mathcal{X} vanishes on D , so there is an invertible \mathcal{X} -twisted sheaf M on D . This proves the claim. \square

Before getting to the main result, we need to extend a standard result about elementary transformations to the case of a simple normal crossings scheme. The case of a regular scheme is handled in [25, Corollary A.7].

Definition 3.6. Suppose Z is a stack and $i : W \subset Z$ is a closed substack. Furthermore, suppose given a quasi-coherent sheaf F on Z and a quotient $q : F|_W \rightarrow Q$. The *elementary transform* of F along q is defined to be the kernel of the morphism $F \rightarrow i_*Q$ induced by the adjunction map and q .

Lemma 3.7. Let X be a scheme and $C \subset X$ an effective Cartier divisor with connected component decomposition $C = \sqcup_i C_i$. Suppose $\pi : \mathcal{X} \rightarrow X$ is a μ_n -gerbe and E is a locally free \mathcal{X} -twisted sheaf. Suppose that $q : E|_C \rightarrow F$ is a surjection to a locally free \mathcal{X} -twisted sheaf F supported on C ; write m_i for the rank of $F|_{C_i}$. Then the determinant of the elementary transform of E along q is isomorphic to

$$\det(E) \otimes \mathcal{O}_X(-\sum_i m_i C_i).$$

Proof. In order for the determinant to be well-defined, we need to check that the subsheaf $G := \ker(E \rightarrow i_*F)$ is perfect when viewed as a complex of \mathcal{X} -twisted $\mathcal{O}_{\mathcal{X}}$ -modules with quasi-coherent cohomology sheaves. In fact, G is locally free. To see this, it suffices to work smooth-locally and prove the following: let Z be a scheme, $i : W \hookrightarrow Z$ an effective Cartier divisor, and \mathcal{E} a locally free sheaf on Z . Given a locally free sheaf \mathcal{F} on W , the kernel of any surjection $\mathcal{E} \rightarrow i_*\mathcal{F}$ is locally free. This in turn reduces to the local case, so we may assume that $Z = \mathrm{Spec} A$, that W is cut out by a single regular element $a \in A$, and that \mathcal{E} and \mathcal{F} are free on Z and W , respectively. Since \mathcal{O}_W has projective dimension 1 over A , so does \mathcal{F} . But since \mathcal{E} has projective dimension 0, it follows that the kernel of any such surjection must also be projective, hence locally free, as desired.

To prove the result it is enough to prove that $\det(i_*F) \cong \mathcal{O}_X(m_1 C_1 + \cdots + m_R C_s)$. Assume henceforth that C is connected, and hence that F has constant rank, say m , everywhere on C . Pulling back to the Severi–Brauer scheme $P \rightarrow X$ associated to the Azumaya algebra $\pi_* \mathcal{E}nd(\mathcal{E})$ (so that $\mathcal{X}|_P$ has trivial Brauer class) and using the fact that $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(P)$ is injective, we are immediately reduced to the analogous statement for trivial Brauer classes. Let L be an invertible \mathcal{X} -twisted sheaf. The classical theory of determinants tells us that $\det(i_*F \otimes L^\vee) \cong \mathcal{O}(mC)$. But the rank of i_*F is 0, so this also computes $\det(i_*F)$, as desired. \square

Proposition 3.8. Let X be a quasi-projective geometrically connected snc surface over a field k of characteristic $p \geq 0$, and let $\mathcal{X} \rightarrow X$ be a μ_n -gerbe, where n is prime to p . Suppose that \mathcal{X} has index n on each irreducible component of X and that the Brauer class of \mathcal{X} vanishes

on each closed subscheme of X of dimension at most 1. (This later condition holds when k is separably closed or X is proper and k is semi-finite by Lemma 3.1.) Then there exists a locally free \mathcal{X} -twisted sheaf of rank n and trivial determinant on X .

Proof. First, we show that if there exists an \mathcal{X} -twisted sheaf F of rank n on X , then F can be chosen to have trivial determinant. Indeed, for $m \gg 0$, we can assume that $\det(F(m)) = \mathcal{O}_Y(D)$, where D is an effective Cartier divisor on X . By choosing an invertible \mathcal{X} -twisted sheaf on D (which is possible by the assumption that the Brauer class of \mathcal{X} vanishes on curves), we can find an invertible quotient Q of $F(m)|_D$. By Lemma 3.7, the elementary transform of $F(m)$ along Q has trivial determinant. Thus, we have constructed a locally free \mathcal{X} -twisted sheaf of rank n on Y with trivial determinant.

Now we proceed by induction on the number of irreducible components of X . If X is irreducible (hence regular), then the existence of a locally free \mathcal{X} -twisted sheaf F of rank n on X follows from the fact that \mathcal{X} has index n and the existence of Azumaya maximal orders over a regular surface. By the above, we can choose F to have trivial determinant.

In general, let $X = X_1 \cup \dots \cup X_r$ be the decomposition of X into its irreducible components. Assume that there exists a locally free \mathcal{X} -twisted sheaf F of rank n on $Y = X_1 \cup \dots \cup X_{r-1}$. Let $C = Y \cap X_r$. By the above, we can choose F with trivial determinant. Consequently the restriction of F to C has trivial determinant, which coincides with the restriction $\mathcal{O}_{X_r}|_C$ of the trivial line bundle from X_r . Hence by Lemma 3.4, there exists a locally free \mathcal{X} -twisted sheaf F_r on X_r such that $F|_C$ is isomorphic to $F_r|_C$. Let E be the fibre product of F and F_r over their restrictions to C (via the chosen isomorphism) in the abelian category of \mathcal{X} -twisted sheaves on X . By applying [29, Theorem 2.1] étale-locally, we see that E is locally free of rank n on X , as desired. By the above, we can choose E with trivial determinant.

By induction, we produce the desired locally free \mathcal{X} -twisted sheaf on X . \square

The following corollary may be found in [25, Corollary 4.2.2.4] in the case when X is smooth over a separably closed field.

Corollary 3.9. *Under the hypotheses of the proposition, the map*

$$H_{\text{ét}}^1(X, \text{PGL}_n) \rightarrow H_{\text{ét}}^2(X, \mu_n)$$

is surjective.

Proof. Given a μ_n -gerbe $\mathcal{X} \rightarrow X$, the proposition produces a locally free \mathcal{X} -twisted sheaf V of rank n . The determinant of V differs from $[\mathcal{X}]$ by a class of $\text{Pic}(X)/n \text{Pic}(X)$. Performing an elementary transformation along a suitable effective Cartier divisor corrects the determinant, by Lemma 3.7. \square

4. FRACKING

In this section we describe a standard trick in deformation theory that kills obstructions in dimension 2. Because it roughly corresponds to “punching holes” in a sheaf, we call this *fracking*.

Lemma 4.1 (Fracking Lemma). *Let R be a regular local ring of dimension at least 2, let F be a free R -module of finite rank prime to the characteristic of the fraction field, and let $f \in \text{Hom}_R(F, F)_0$ be non-zero at the closed point of $\text{Spec } R$, where $\text{Hom}_R(F, F)_0$ denotes the submodule of $\text{Hom}_R(F, F)$ of trace-zero endomorphisms. Then there exists a submodule $G \subseteq F$ such that F/G is supported at the closed point and such that f is not in the image of the natural inclusion*

$$p : \text{Hom}_R(G, G)_0 \rightarrow \text{Hom}_R(F, F)_0$$

induced by taking the reflexive hull of G .

Proof. Let \mathfrak{m} be the maximal ideal of R . Because f has trace zero and non-zero scalar matrices all have non-zero trace (by the assumption that F has rank prime to the characteristic of the fraction field of R), there is a line \overline{L} in $\overline{F} = F/\mathfrak{m}$ such that f does not preserve \overline{L} (i.e., $f(\overline{L})$ is not contained in \overline{L}). Let G be the kernel of $F \rightarrow \overline{F}/\overline{L}$. Consider the submodule B of

$\mathrm{Hom}_R(F, F)_0$ of traceless endomorphisms that preserve the flag $\overline{L} \subseteq \overline{F}$ at the closed point. Any traceless endomorphism of G preserves this flag. Indeed, if $g \in \mathrm{Hom}_R(G, G)_0$, then there is a commutative diagram

$$\begin{array}{ccc} \overline{G} & \longrightarrow & \overline{F} \\ \downarrow g & & \downarrow p(g) \\ \overline{G} & \longrightarrow & \overline{F}. \end{array}$$

By construction, the image of each horizontal map is the line \overline{L} in \overline{F} . Hence, $p(g)$ preserves \overline{L} so that p factors through the inclusion $B \rightarrow \mathrm{Hom}_R(F, F)_0$. On the other hand, there is an exact sequence

$$0 \rightarrow B \rightarrow \mathrm{Hom}_R(F, F)_0 \rightarrow \mathrm{Hom}_R(\overline{L}, \overline{F}/\overline{L}).$$

Since the map f we start with is non-zero on the right, it is not contained in B , and hence is not in the image of p . \square

5. DEFORMATION THEORY OF PERFECT TWISTED SHEAVES

The material in this section is similar to [24], except our infinitesimal deformations of the base scheme are not flat. We review the theory in this case; there are no essential differences.

Let $i : X_0 \subseteq X$ be a closed subscheme of a quasi-separated noetherian scheme X defined by a square-zero sheaf of ideals I of \mathcal{O}_X . Let \mathcal{F} be a complex of \mathcal{O}_{X_0} -modules in $D_{\mathrm{qc}}(X_0)$.

Definition 5.1. A *deformation* of \mathcal{F}_0 to X consists of a complex \mathcal{F} in $D_{\mathrm{qc}}(X)$ and a quasi-isomorphism $\mathcal{O}_{X_0} \otimes_{\mathcal{O}_X}^L \mathcal{F} \simeq \mathcal{F}_0$.

Lemma 5.2. *If \mathcal{F}_0 is perfect and \mathcal{F} is a deformation of \mathcal{F}_0 to X , then \mathcal{F} is perfect.*

Proof. Note that there is a distinguished triangle $i_* I \otimes_{\mathcal{O}_X}^L \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* \mathcal{F}_0$ in $D_{\mathrm{qc}}(X)$. If we prove that \mathcal{F} has finite Tor-amplitude, then it will follow from [35, Theorem 2.5.5] that $i_* I \otimes_{\mathcal{O}_X}^L \mathcal{F}$ is quasi-isomorphic to

$$i_* \left(I \otimes_{\mathcal{O}_{X_0}}^L \mathrm{Li}^* \mathcal{F} \right) \simeq i_* (I \otimes_{\mathcal{O}_{X_0}} \mathcal{F}_0).$$

On a quasi-separated noetherian scheme, such as X , the perfect complexes of \mathcal{O}_X -modules are precisely those complexes which have coherent cohomology sheaves and which moreover have bounded Tor-amplitude. See [35, Example 2.2.8 and Proposition 2.2.12]. Since $i_* \mathcal{F}_0$ and $i_* (I \otimes_{\mathcal{O}_{X_0}} \mathcal{F}_0)$ have coherent cohomology sheaves, it follows that if we show that \mathcal{F} has finite Tor-amplitude, the lemma will follow.

Recall that a complex \mathcal{F} of $D_{\mathrm{qc}}(X)$ has Tor-amplitude contained in an interval with integer endpoints $[a, b]$ if and only if $\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) = 0$ for all \mathcal{O}_X -modules \mathcal{G} and all $n \notin [a, b]$. Suppose that \mathcal{F}_0 has Tor-amplitude contained in $[a, b]$, and let \mathcal{F} be a deformation of \mathcal{F}_0 to X . Suppose that $n \in \mathbf{Z}$, and suppose that $\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is not zero. Then, there is a closed point x of X such that $\mathcal{T}or_n^{\mathcal{O}_X}(k(x), \mathcal{F})$ is not zero. But,

$$\mathcal{F} \otimes_{\mathcal{O}_X}^L k(x) \simeq \mathcal{F} \otimes_{\mathcal{O}_X}^L (\mathcal{O}_{X_0} \otimes_{\mathcal{O}_{X_0}}^L k(x)) \simeq \mathcal{F}_0 \otimes_{\mathcal{O}_{X_0}}^L k(x).$$

Hence,

$$\mathcal{T}or_n^{\mathcal{O}_X}(k(x), \mathcal{F}) \cong \mathcal{T}or_n^{\mathcal{O}_{X_0}}(k(x), \mathcal{F}_0),$$

which implies that $n \in [a, b]$, as desired. \square

Definition 5.3. Recall that the *determinant* of a perfect complex \mathcal{F} is computed locally as

$$\det(\mathcal{F}) = \bigotimes_{n \in \mathbf{Z}} \det(C^n)^{(-1)^n}$$

where $\mathcal{F} \simeq C^\bullet$, a complex of locally free finite rank vector bundles on X .

Definition 5.4. Suppose that \mathcal{F} has $\det(\mathcal{F}_0) \cong \mathcal{O}_{X_0}$, and fix one such trivialization. An *equideterminantal deformation* of \mathcal{F}_0 is a deformation \mathcal{F} as above together with a deformation of the trivialization of the determinant.

The next proposition is well-known to experts and follows immediately from the techniques in [17].

Proposition 5.5. *Let $X_0 \hookrightarrow X$ be a closed subscheme of a quasi-separated noetherian scheme X defined by a square-zero sheaf of ideals I of \mathcal{O}_X . Let \mathcal{F} be a perfect complex of \mathcal{O}_{X_0} -modules. Then, the obstruction to the existence of an equideterminantal deformation of \mathcal{F}_0 to X lies in $\mathbf{H}^2(X_0, I \otimes_{\mathcal{O}_{X_0}}^{\mathbf{L}} s\mathbf{R}\mathcal{E}nd(\mathcal{F}))$, where $s\mathbf{R}\mathcal{E}nd(\mathcal{F})$ denotes the traceless part of the complex of endomorphisms.*

We will use these methods for \mathcal{X} -twisted sheaves later. The arguments are the same.

6. ALTERATIONS

In this section, we use Gabber's theory of prime-to- ℓ alterations over a discrete valuation ring; see [18], [19]. For an example of the statement we are interested in, see [8, Théorème 3.25].

Definition 6.1. Let ℓ be a prime number and X a scheme of finite type over an excellent ring. An ℓ' -alteration $X' \rightarrow X$ is a proper surjective generically finite map such that for every maximal point η of X , there exists a maximal point η' of X' over η such that the residue field extension $\kappa(\eta')/\kappa(\eta)$ has degree prime to ℓ .

Lemma 6.2. *Let X be an integral scheme, $X' \rightarrow X$ an ℓ' -alteration, η' a maximal point of X' dominating X , and $\alpha \in \mathrm{Br}(\kappa(X))$. If $\mathrm{ind}(\alpha_{\kappa(\eta')}) = \ell^N$ then $\mathrm{ind}(\alpha) = \ell^N$.*

Proof. Because $\kappa(\eta')/\kappa(X)$ has degree prime-to- ℓ , the result follows by a standard restriction-corestriction argument. \square

Definition 6.3. Let R be a discrete valuation ring, $s \in \mathrm{Spec} R$ the closed point, and X an R -scheme. If X is equidimensional, flat and of finite type over R , the generic fibre of X over $\mathrm{Spec} R$ is smooth, and the reduced special fibre $X_{0,\mathrm{red}}$ is a simple normal crossings divisor on X , then X is said to be *quasi-semistable* over R .

The following two results are a distillation of the main results of Gabber's theory of uniformization by ℓ' -alterations; cf. See [18, Theorem 1.4], [19, X.2], and [8, Théorème 3.25].

Lemma 6.4. *If X is quasi-semistable over R , then $X \rightarrow \mathrm{Spec} R$ is étale locally of the form*

$$(6.1) \quad X = \mathrm{Spec} R[t_1, \dots, t_n] / (t_1^{a_1} \cdots t_r^{a_r} - \pi),$$

where π is a uniformizing parameter of R .

Theorem 6.5 (Gabber). *If R is an excellent henselian discrete valuation ring with residue field k of characteristic $p \geq 0$ and fraction field K , and X is a proper scheme over R , then for any prime $\ell \neq p$, there exists a commutative diagram of ℓ' -alterations*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec} R & \longleftarrow & \mathrm{Spec} R' \end{array}$$

such that X' is regular and quasi-semistable and projective over R' , an excellent henselian discrete valuation ring.

Proposition 6.6. *Let R be an excellent henselian discrete valuation ring, X be an integral scheme proper over R of relative dimension d , and fix $\alpha \in \mathrm{Br}(\kappa(X))[\ell]$. Then there exists a diagram of morphisms*

$$(6.2) \quad \begin{array}{ccccccc} X & \xleftarrow{f} & X' & \xleftarrow{g} & Y & \xleftarrow{h} & Y' \\ \downarrow & & \searrow & & \swarrow & & \downarrow \\ \mathrm{Spec} R & \longleftarrow & & \mathrm{Spec} R' & \longleftarrow & & \mathrm{Spec} R'' \end{array}$$

where

- (1) R'/R and R''/R' are finite extensions of excellent henselian discrete valuation rings,
- (2) f and h are ℓ' -alterations,
- (3) X' and Y' are regular and integral,
- (4) $X' \rightarrow \operatorname{Spec} R'$ and $Y' \rightarrow \operatorname{Spec} R''$ are projective and quasi-semistable,
- (5) Y is integral and the function field extension induced by g has the form $\kappa(Y) = \kappa(X')(\sqrt[\ell]{f_1}, \dots, \sqrt[\ell]{f_N})$ for some N , and
- (6) $\alpha_{\kappa(Y')}$ lies in the subgroup $\operatorname{Br}(Y')[\ell]$.

Moreover, if there exists an ℓ -Pirutka $n \times (d+1)$ matrix then we may take $N = n$.

Proof. By Theorem 6.5, there exists a commutative diagram of ℓ' -alterations

$$\begin{array}{ccc} X & \xleftarrow{f_1} & X_1 \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} R_1 \end{array}$$

where X_1 is regular and integral, and projective over R_1 . Consider the ramification divisor D_1 of $\alpha_{\kappa(X')}$ on X_1 . By an application of Gabber's embedded uniformization (see [18, Theorem 1.4]), there exists a further commutative diagram of ℓ' -alterations

$$\begin{array}{ccc} X_1 & \xleftarrow{f_2} & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec} R_1 & \longleftarrow & \operatorname{Spec} R' \end{array}$$

where X' is regular and integral, and projective and quasi-semistable over R' , and such that the ramification divisor D of $\alpha_{\kappa(X')}$ has normal crossings. We may need to blow up to arrange that the ramification divisor has strict normal crossings, cf. [9, Section 2.4]. We compose these two squares to arrive at the left most square in the desired diagram.

By Theorem 2.3.9 and Example 2.4.1, there exist rational functions f_1, \dots, f_N in $\kappa(X')$ such that α_L is unramified, where $L = \kappa(X')(\sqrt[\ell]{f_1}, \dots, \sqrt[\ell]{f_N})$. (We can always choose $N = (d+1)^2$, and moreover, if there exists an ℓ -Pirutka $n \times (d+1)$ matrix, we can take $N = n$.) Let Y be the normalization of X' in L and $g : Y \rightarrow X'$ the induced map. We now apply Theorem 6.5 again, to arrive at an ℓ' -alteration

$$\begin{array}{ccc} Y & \xleftarrow{h} & Y' \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longleftarrow & \operatorname{Spec} R'' \end{array}$$

where Y' is regular and integral, and projective and quasi-semistable over R'' . Since $\alpha_{\kappa(Y)}$ is unramified, the same holds for $\alpha_{\kappa(Y')}$, and hence by purity, $\alpha_{\kappa(Y')} \in \operatorname{Br}(Y')$. \square

7. PROOFS OF THEOREM 1.1 AND THEOREM 1.4

We may, first of all, assume that $\operatorname{per}(\alpha) = \ell$ a prime distinct from the residue characteristics of X . By Theorem 6.5, Proposition 6.6, Theorem 2.3.9, and the results of Section 2.4, the proofs of Theorem 1.1 and Theorem 1.4 both reduce to proving Theorem 1.4 under the additional hypothesis that $X \rightarrow \operatorname{Spec} R$ is quasi-semistable. Let $\mathcal{X} \rightarrow X$ be the μ_ℓ -gerbe associated to α .

Let π be a uniformizing parameter of R , so that (π) denotes the sheaf of ideals in \mathcal{O}_X that cuts out the special fibre X_1 , and let $I \supseteq (\pi)$ be the sheaf of ideals in \mathcal{O}_X that cuts out the *reduced* special fibre $X_0 \subseteq X_1$. By Proposition 3.8, there exists an \mathcal{X} -twisted sheaf F of rank ℓ on X_0 with trivial determinant. To finish the proof, it suffices to find a perfect twisted subsheaf $G \subset F$ such that $\operatorname{rank}(G) = \operatorname{rank}(F)$ and G deforms to an \mathcal{X} -twisted sheaf over the formal scheme \widehat{X} . Indeed, by the Grothendieck Existence Theorem [14, Théorème 5.1.4],

any such formal deformation algebraizes to yield an \mathcal{X} -twisted sheaf of rank ℓ on $X_{\hat{R}}$, the pullback of $X \rightarrow \operatorname{Spec} R$ to the completion \hat{R} of R . Arguing as in [10, Section 6], it follows that $\operatorname{ind}(\alpha_{\kappa(X)})$ divides ℓ , completing the proof.

The rest of this section is devoted to producing the desired formal deformation. We will do this by analyzing the formal local structure of X near X_0 and then applying Lemma 4.1 to eliminate obstructions to deforming across infinitesimal neighborhoods of X_0 .

Given two sheaves of ideals \mathcal{I} and \mathcal{J} on a scheme Y , define

$$\mathcal{I} \diamond \mathcal{J} = (\mathcal{I} \mathcal{J} : \mathcal{I} \cap \mathcal{J}).$$

If $f \in \Gamma(Y, \mathcal{O}_Y)$ is an everywhere regular section, then we have $(f\mathcal{I} : f\mathcal{J}) = f(\mathcal{I} : \mathcal{J})$ and thus $f\mathcal{I} \diamond f\mathcal{J} = f(\mathcal{I} \diamond \mathcal{J})$. If Y is the spectrum of a UFD, then for any two sections $f, g \in \Gamma(Y, \mathcal{O}_Y)$, we have

$$(7.1) \quad (f) \diamond (g) = (\gcd(f, g))$$

Since this can be checked locally, it also follows that (7.1) holds in any locally factorial scheme.

Since $I/(\pi)$ is nilpotent, there is a least m such that $I^m \subseteq (\pi)$. Given $1 \leq a \leq m$ and $b \geq 0$, let $J_{a,b} = I^a(\pi^b) \diamond (\pi^{b+1})$. The ideals $J_{a,b}$ have the following properties.

- (1) $J_{a+1,b} \subseteq J_{a,b}$ for $1 \leq a \leq m-1$ and $J_{1,b+1} \subseteq J_{m,b}$.
- (2) $J_{m,b} = (\pi^{b+1})$. Indeed, $I^m(\pi^b) \subseteq (\pi^{b+1})$, so that $(\pi^{b+1}) \subseteq I^m(\pi^b) \diamond (\pi^{b+1})$. Since X is regular, I is locally principal, so that the inclusion $(\pi^{b+1}) \subseteq J_{m,b}$ is locally an equality and hence an equality.
- (3) $J_{a,b}/J_{a+1,b} \cong J_{a,0}/J_{a+1,0}$ for $1 \leq a \leq (m-1)$ and $J_{m,b}/J_{1,b+1} \cong J_{m,0}/J_{1,1} \cong \mathcal{O}_X/I$.

This also follows from the fact that π is a regular section of \mathcal{O}_X .

Consider the filtration

$$(7.2) \quad \begin{aligned} I &= J_{1,0} \supset J_{2,0} \supset \dots \supset J_{m-1,0} \supset (\pi) = J_{m,0} \supset \\ &J_{1,1} \supset J_{2,1} \supset \dots \supset J_{m-1,1} \supset (\pi^2) = J_{m,1} \supset \\ &J_{1,2} \supset J_{2,2} \supset \dots \supset J_{m-1,2} \supset (\pi^3) = J_{m,2} \supset \\ &\dots \end{aligned}$$

By the above calculations, there are only finitely many \mathcal{O}_X -modules appearing in the list of successive quotients in this filtration. By our choice of m , all of the successive quotients are non-zero. Moreover, multiplication by I kills any of these \mathcal{O}_X -modules, so we can view them as \mathcal{O}_{X_0} -modules.

Claim. *Each successive quotient in the filtration defined in (7.2) is locally free of rank 1 on its support, which consists of the union of a set of components of X_0 .*

The claim is immediate for $J_{m,b}/J_{1,b+1} \cong \mathcal{O}_X/I = \mathcal{O}_{X_0}$. For $1 \leq a < m$, we verify the claim étale locally, where we can appeal to the étale local structure (6.1) of X . Thus, we may assume that our regular scheme is $X = \operatorname{Spec} R[t_1, \dots, t_n]/(t_1^{a_1} \dots t_r^{a_r} - \pi)$. In this case, $I = (t_1 \dots t_r)$ and $(\pi) = (t_1^{a_1} \dots t_r^{a_r})$. Using (7.1), we find that

$$J_{a,0} = (t_1^{\min(a,a_1)} \dots t_r^{\min(a,a_r)}),$$

and hence the quotient $J_{a,0}/J_{a+1,0}$ is isomorphic to

$$\mathcal{O}_X/(t_1^{\epsilon(1,a)} \dots t_r^{\epsilon(r,a)}),$$

where

$$\epsilon(i, a) = \begin{cases} 0 & \text{if } a_i \leq a, \\ 1 & \text{if } a_i > a. \end{cases}$$

Note that, by our choice of m , for any $1 \leq a < m$, some $\epsilon(i, a)$ is non-zero. It follows that the successive quotient is (étale locally) isomorphic to the structure sheaf of some collection of components of the reduced special fibre, proving the claim.

For notational simplicity, set $M_0 = \mathcal{O}_{X_0}$ and $M_i = J_{i,0}/J_{i+1,0}$ for $1 \leq i < m$. We claim that there is a perfect subsheaf $G \subseteq F$ such that $\text{rank}(G) = \text{rank}(F)$ and

$$\text{Ext}_{X_0}^2(G, M_i \otimes G)_0 = 0$$

for $0 \leq i < m$. If this is so then the obstruction of Proposition 5.5 to deforming any such G through the filtration (7.2) vanishes, giving the desired formal deformation.

To prove the required vanishing, it is sufficient to prove that

$$\prod_{i=1}^{m-1} \text{Ext}_{X_0}^2(G, M_i \otimes G)_0^\vee = 0.$$

Since X_0 is Gorenstein (with invertible dualizing sheaf ω_X), for any sheaf L whose rank at each generic point of X_0 is invertible in the base field there is a natural isomorphism

$$\bigoplus_{i=0}^{m-1} \text{Hom}_{X_0}(M_i \otimes L, \omega_X \otimes L)_0 \rightarrow \prod_{i=0}^{m-1} \text{Ext}_{X_0}^2(L, M_i \otimes L)_0^\vee.$$

Let $f_1, \dots, f_n : M_i \otimes F \rightarrow \omega_X \otimes F$ be a basis for the group $\bigoplus_{i=0}^{m-1} \text{Hom}_{X_0}(M_i \otimes F, \omega_X \otimes F)_0$. Choose distinct regular closed points x_1, \dots, x_n of X_0 such that f_j is non-zero at x_j . This is possible since each M_i is a pushforward of a line bundle on a set of components and since F is locally free on X_0 . In particular, x_j is a closed point of a component of X_0 away from the rest of the components. By applying Lemma 4.1 at each of the points x_j , we obtain a subsheaf $G \subseteq F$ such that

- (1) G is a perfect sheaf with reflexive hull F ,
- (2) the maps

$$\text{Hom}_{X_0}(M_i \otimes G, \omega_X \otimes G)_0 \rightarrow \text{Hom}_{X_0}(M_i \otimes F, \omega_X \otimes F)_0$$

induced by passing to reflexive hulls are injective, but

- (3) no f_j is in the image.

It follows that

$$\prod_{i=1}^{m-1} \text{Ext}_{X_0}^2(G, M_i \otimes G)_0^\vee = 0,$$

concluding the proof.

Remark 7.1. In the proof, we may have to take a torsion free subsheaf of F in order to remove obstructions to deforming off of X_0 . In that case, the resulting \mathcal{X} -twisted sheaf may not be locally free. Moreover, a reflexive sheaf on a regular threefold need not be locally free, though it will have torsion free fibres over R . Algebraically speaking, this process may yield a maximal order in the division algebra corresponding to α that is not locally free (cf. [1]).

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